New counterterms induced by trans-Planckian physics in semiclassical gravity

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We consider free and self-interacting quantum scalar fields satisfying modified dispersion relations in the framework of Einstein-Aether theory. Using adiabatic regularization, we study the renormalization of the equation for the mean value of the field in the self-interacting case, and the renormalization of the semiclassical Einstein-Aether equations for free fields. In both cases we consider Bianchi type I background spacetimes. Contrary to what happens for *free* fields in *flat* Robertson-Walker spacetimes, the self-interaction and/or the anisotropy produce non-purely geometric terms in the adiabatic expansion, i.e terms that involve both the metric $g_{\mu\nu}$ and the aether field u_μ. We argue that, in a general spacetime, the renormalization of the theory would involve new counterterms constructed with $g_{\mu\nu}$ and u_μ, generating a fine-tuning problem for the Einstein-Aether theory.

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I. INTRODUCTION

In the last years it has been realized that the (still unknown) physics at very high energies may not be inaccessible from an observational point of view. Indeed, trans-Planckian physics may have left an imprint in the inhomogeneities of the cosmic microwave background radiation [1], in the evolution of the scale factor of the universe [2], in the propagation of gamma ray bursts [3], etc.

In the absence of a full theory, the theoretical approach to this problem is phenomenological. One possibility, that we will consider here, is to assume that the physics at high energies is such that its main effect is a modification of the dispersion relation of the quantum fields, thus violating Lorentz symmetry. Although this is a simplistic approach, it could be useful to investigate whether the trans-Planckian effects could lead to observable consequences or not in a given particular situation, by testing the robustness of the results under changes in the dispersion relations at very high energies.

The Modified Dispersion Relations (MDR) will obviously affect the structure of the quantum field theory, in particular its renormalizability. Having in mind applications to cosmology, in previous papers [4, 5, 6], we have analyzed in detail the renormalization of free field theories with MDR in flat Robertson Walker spacetimes. We have shown that the theory can be renormalized using a generalization of the well known adiabatic regularization [7, 8] that is used in theories with standard dispersion relations. As for the usual case, the adiabatic expansion of the energy momentum tensor contains divergent terms that can be written in terms of geometric tensors in n-dimensions, and therefore the theory can be renormalized by absorbing the infinities into the bare gravitational constants of the theory. It is remarkable that this can be done whatever the dispersion relation. This somewhat surprising result could be a peculiarity of flat Robertson Walker metrics [9] and/or valid only for free fields, and therefore it is of interest to investigate more general situations.

In this paper we extend the adiabatic regularization to the case of self-interacting fields and anisotropic metrics (Bianchi type I). We will work within the context of the so called Einstein-Aether theory [10], a covariant theory of gravity in which the metric is coupled to a dynamical vector field. This field, that breaks

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Lorentz invariance dynamically, is also coupled to the derivatives of the quantum matter fields, leading to MDR that contain higher powers of the momenta. The specific model is introduced in Section II.

In Section III we consider a self-interacting scalar field on Bianchi type I metrics and discuss the renormalization of the equation for the mean value of the field ϕ_0. In order to do this, it will be necessary to compute the mean value of the fluctuations of the field $\langle\hat{\phi}^2\rangle = \langle(\phi - \phi_0)^2\rangle$. We will calculate explicitly this quantity up to the second adiabatic order and show that, contrary to what happens for the usual dispersion relation, the second adiabatic order cannot be entirely written in terms of the metric and its derivatives, but also involve the aether field u_μ and its derivatives. This property of $\langle\hat{\phi}^2\rangle$ is valid even for free fields in flat Robertson-Walker spacetimes.

In Section IV we analyze the renormalizability of the Semiclassical Einstein-Aether Equations (SEAE) for the case of free scalar fields with MDR in Bianchi type I universes. We compute $\langle T_{\mu\nu}\rangle$ up to the second adiabatic order. The zeroth adiabatic order is divergent whatever the dispersion relation. Being proportional to $g_{\mu\nu}$, the divergence can be absorbed into a redefinition of the comological constant. The second adiabatic order is shown to be divergent for dispersion relations that involve powers of the momenta smaller than or equal to four. This adiabatic order contains a term proportional to $G_{\mu\nu}$, that renormalizes Newton's constant. However, it also contains an additional non-purely geometric term, proportional to the variation of $(\nabla_\mu u^\mu)^2$. When this term is divergent, a new counterterm has to be introduced to renormalize the theory, even if originally not present in the classical Lagrangian. On the other hand, if it is finite, a counterterm would be necessary to make the theory consistent with observations.

In Section V we argue that, for a general metric, the renormalization of the infinities produced by a quantum free field satisfying MDR will induce all possible counterterms involving up to two derivatives of the metric $g_{\mu\nu}$ and the vector u_μ. As shown in Ref. [11], the coefficients of terms like $(\nabla_\mu u^\mu)^2$, $R_{\mu\nu}u^\mu u^\nu$, etc, are strongly constrained observationally by post-Newtonian parameters, and therefore the counterterms induced by trans-Planckian physics should be fine tuned to satisfy these constraints.

Throughout the paper we set $c = 1$ and adopt the sign convention denoted (+++) by Misner, Thorne, and Wheeler [12].

II. THE MODEL

We work in the frame of a generally covariant theory of gravity coupled to a dynamical vector field u^μ that breaks local Lorentz symmetry. The most general action that is quadratic in derivatives is given by [10]:

$$S_G = \frac{1}{16\pi G}\int d^n x\sqrt{-g}(R - 2\Lambda + \mathcal{L}_u), \tag{1}$$

where $g = det(g_{\mu\nu})$, R is the Ricci scalar, Λ and G are the bare cosmological and Newton's constants, and $\mathcal{L}_u$ describe the dynamics of the additional degree of freedom u^μ,

$$\mathcal{L}_u = -\tilde{\lambda}(g^{\mu\nu}u_\mu u_\nu + 1) - b_1 F_{\mu\nu}F^{\mu\nu} - b_2(\nabla_\mu u^\mu)^2 - b_3 R_{\mu\nu}u^\mu u^\nu - b_4 u^\rho u^\sigma \nabla_\rho u_\mu \nabla_\sigma u^\mu, \tag{2}$$

where $F_{\mu\nu} = \nabla_\mu u_\nu - \nabla_\nu u_\mu$. The Lagrange multiplier $\tilde{\lambda}$ is introduced to impose the condition $u_\mu u^\mu = -1$ and the coefficients b_i $(i = 1, 2, 3, 4)$ are arbitrary. The term $\nabla_\mu u_\nu \nabla^\nu u^\mu$ coincides with $(\nabla_\mu u^\mu)^2 - R_{\mu\nu}u^\mu u^\nu$ up to a total derivative, and hence has been omitted.

We consider a quantum scalar field ϕ with a generalized dispersion relation propagating in a curved space-time with a classical background metric given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \equiv -(u_\mu dx^\mu)^2 + \perp_{\mu\nu} dx^\mu dx^\nu, \tag{3}$$

where $\mu, \nu = 0, 1...n-1$ (with n the space-time dimension) and $\perp_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu$. The action for the scalar field can be written as:

$$S_\phi = \int d^n x\sqrt{-g}(\mathcal{L}_\phi + \mathcal{L}_{cor} + \mathcal{L}_{int}), \tag{4}$$

where $\mathcal{L}_\phi$ is the standard Lagrangian of a free, massive, minimally coupled scalar field

$$\mathcal{L}_\phi = -\frac{1}{2}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right], \tag{5}$$

$\mathcal{L}_{cor}$ is the corrective lagrangian that gives rise to a generalized dispersion relation

$$\mathcal{L}_{cor} = -\sum_{s,p} b_{sp}(\mathcal{D}^{2s}\phi)(\mathcal{D}^{2p}\phi), \tag{6}$$

where $0 < p \leq s$, b_{sp} are arbitrary coefficients, and $\mathcal{D}^2\phi \equiv \perp_\mu^\lambda \nabla_\lambda \perp_\gamma^\mu \nabla^\gamma\phi$ (∇_μ is the covariant derivative corresponding to the metric $g_{\mu\nu}$ and $\perp_\mu^\lambda \equiv g^{\lambda\nu}\perp_{\mu\nu}$). The interaction Lagrangian $\mathcal{L}_{int}$ contains the following terms:

$$\mathcal{L}_{int} = -\frac{1}{2}[\xi R + \xi_1 F_{\mu\nu}F^{\mu\nu} + \xi_2(\nabla_\mu u^\mu)^2 + \xi_3\nabla_\mu u_\nu\nabla^\nu u^\mu + \xi_4 u^\rho u^\sigma\nabla_\rho u_\mu\nabla_\sigma u^\mu + \xi_5 u^\mu u^\nu R_{\mu\nu}]\phi^2 - \lambda\phi^4, \tag{7}$$

where ξ, ξ_i ($i = 1,2,3,4,5$) and λ are bare parameters. Note that, in addition to the self-interaction and the standard coupling to the Ricci scalar, we have also included couplings between ϕ^2 and non-purely geometric terms that involve the aether field u_μ. Note also that, if we assume that the MDR depart from the usual one at a given scale M_C, the coefficients b_{sp} scale as $b_{sp} \sim M_C^{2(1-s-p)}$.

In the rest of the paper we will consider a four-dimensional Bianchi type I space-time with line element

$$ds^2 = -dt^2 + \sum_{i=1}^{3} C_i(t)dx_i^2 = -C(\eta)d\eta^2 + \sum_{i=1}^{3} C_i(t)dx_i^2, \tag{8}$$

where $C = (C_1C_2C_3)^{1/3}$, $d\eta = dt/C^{1/2}$, and $u_\mu \equiv C^{1/2}(\eta)\delta_\mu^\eta$. Therefore, in this frame $F_{\mu\nu} = 0$ and $u^\mu\nabla_\mu u_\nu = 0$. In what follows we use primes for denoting derivatives with respect to the conformal time η. No sum convention in spatial (latin) indices is assumed. The generalized dispersion relation takes the form

$$\omega_k^2 = C(\eta)\left[m^2 + x + 2\sum_{s,p}(-1)^{s+p}\, b_{sp}\, x^{(s+p)}\right], \tag{9}$$

where $x = \sum_{i=1}^3 k_i^2/C_i \equiv \sum_{i=1}^3 x_i \equiv \sum_{i=1}^3 x\lambda_i^2$, with $\sum_{i=1}^3 \lambda_i^2 = 1$.

III. SELF-INTERACTING SCALAR FIELD IN BIANCHI TYPE I SPACE-TIMES

In this section we are concerned with the renormalization of the equation of motion for the expectation value of a self-interacting scalar field ($\lambda \neq 0$) propagating in a four-dimensional Bianchi type I space-time. We assume that the state of the system is such that the expectation value of the field is ϕ_0. Then, defining a new quantum field $\hat{\phi}$ as $\phi = \phi_0 + \hat{\phi}$, the equation of motion for ϕ_0 in the one-loop approximation is given by

$$\Box\phi_0 - \left[m^2 + \xi R + \xi_2(\nabla_\mu u^\mu)^2 + \xi_3\nabla_\mu u_\nu\nabla^\nu u^\mu + \xi_5 R_{\mu\nu}u^\mu u^\nu + 2\sum_{s,p\leq s} b_{sp}\mathcal{D}^{2(s+p)} + 12\lambda\langle\hat{\phi}^2\rangle\right]\phi_0 - 4\lambda\phi_0^3 = 0. \tag{10}$$

The Fourier modes of the scaled field $\chi = C^{1/2}\hat{\phi}$ satisfy

$$\chi_k'' + \left[\omega_k^2 + \left(\xi - \frac{1}{6}\right)CR + Q + \xi_2 C(\nabla_\mu u^\mu)^2 + \xi_3 C\nabla_\mu u_\nu\nabla^\nu u^\mu + \xi_5 CR_{\mu\nu}u^\mu u^\nu + 12C\lambda\phi_0^2\right]\chi_k = 0, \tag{11}$$

with the usual normalization condition

$$\chi_k {\chi'_k}^* - \chi'_k \chi_k^* = i \; . \tag{12}$$

The explicit expressions for the different terms in Eqs. (10) and (11) are, in Bianchi type I metrics,

$$(\nabla_\mu u^\mu)^2 = \frac{9D^2}{4C}, \tag{13a}$$

$$R_{\mu\nu}u^\mu u^\nu = -\frac{3}{C}\left[\frac{D'}{2} + 2Q\right], \tag{13b}$$

$$R = \frac{1}{C}\left[3D' + \frac{3}{2}D^2 + 6Q\right], \tag{13c}$$

$$\nabla_\mu u_\nu \nabla^\nu u^\mu = \sum_{i=1}^{3} \frac{d_i^2}{4C} = \frac{3}{4C}(D^2 + 8Q), \tag{13d}$$

$$Q = \frac{1}{72}\sum_{i<j}^{3}(d_i - d_j)^2, \tag{13e}$$

where $d_i = C'_i/C_i$ and $D = \sum_{i=1}^3 d_i/3 = C'/C$. Note that for the metric we are considering

$$2R_{\mu\nu}u^\mu u^\nu + R = (\nabla_\mu u^\mu)^2 - \nabla_\mu u_\nu \nabla^\nu u^\mu, \tag{14}$$

and therefore without loss of generality we can set $\xi_5 = 0$.

For dispersion relations such that the mean value $\langle\hat{\phi}^2\rangle$ in Eq. (10) is divergent, the infinities must be absorbed into the bare constants of the theory. To implement the renormalization, we start by expressing the field modes χ_k in the well known form

$$\chi_k = \frac{1}{\sqrt{2W_k}} \exp\left(-i\int^\eta W_k(\tilde{\eta})d\tilde{\eta}\right), \tag{15}$$

which allows us to write

$$\langle\hat{\phi}^2\rangle = \frac{1}{(2\pi)^3 C}\int d^3k |\chi_k|^2 = \frac{1}{(2\pi)^3 C}\int d^3k \frac{1}{2W_k}. \tag{16}$$

Substitution of Eq. (15) into Eq. (11) yields

$$W_k^2 = \omega_k^2 + \left(\xi - \frac{1}{6}\right)CR + Q + \xi_2 C(\nabla_\mu u^\mu)^2 + \xi_3 C\nabla_\mu u_\nu \nabla^\nu u^\mu + 12\lambda C\phi_0^2 + \frac{5}{16}\frac{[(W_k^2)']^2}{W_k^4} - \frac{1}{4}\frac{(W_k^2)''}{W_k^2}. \tag{17}$$

For adiabatic regularization we need the approximate solution of this non-linear differential equation that is obtained by assuming that W_k^2 is a slowly varying function of η. In this adiabatic or WKB approximation the adiabatic order of a term is given by the number of time derivatives of the metric plus the power of ϕ_0 [13]. The WKB approximation can be obtained by solving the Eq.(17) iteratively

$$W_k = {}^{(0)}W_k + {}^{(2)}W_k + \ldots \, , \tag{18}$$

where the superscript denote the adiabatic order. To lowest order we have ${}^{(0)}W_k = \omega_k$. The second adiabatic order can be computed replacing W_k by ω_k on the right-hand side of Eq. (17). Thus, we straightforwardly obtain

$$\begin{aligned} {}^{(2)}W_k^2 \; = \; & CR(\xi - \frac{1}{6}) + Q + \frac{D^2}{16} - \frac{D'}{4} + \xi_2 C(\nabla_\mu u^\mu)^2 + \xi_3 C\nabla_\mu u_\nu \nabla^\nu u^\mu + 12\lambda C\phi_0^2 \\ & - \; \frac{(f+1)}{4}\sum_{i=1}^{3}\lambda_i^2\left[\frac{Dd_i}{2} + d_i^2 - d'_i\right] + \frac{1}{16}\left(\sum_{i=1}^{3} d_i\lambda_i^2\right)^2\left[f^2 + 6f - 4\dot{f} + 5\right], \end{aligned} \tag{19}$$

where we have defined the function

$$f \equiv \frac{d \ln \tilde{\omega}_k^2}{d \ln x} - 1, \tag{20}$$

with $\tilde{\omega}_k^2 \equiv \omega_k^2/C$. We have also used that

$$\frac{(\omega_k^2)'}{\omega_k^2} = D - (f+1)\sum_{i=1}^{3} d_i \lambda_i^2, \tag{21a}$$

$$\frac{(\omega_k^2)''}{\omega_k^2} = D' + D^2 + (f+1)\sum_{i=1}^{3} \lambda_i^2 [d_i^2 - 2 d_i D - d_i'] + (\dot{f} + f^2 + f)\left(\sum_{i=1}^{3} d_i \lambda_i^2\right)^2, \tag{21b}$$

where a dot indicates a derivative with respect to $\ln x$.

We proceed as for the standard dispersion relation, defining the renormalized expectation value as

$$\langle \hat{\phi}^2 \rangle_{ren} = \langle \hat{\phi}^2 \rangle - \langle \hat{\phi}^2 \rangle_{ad2}, \tag{22}$$

with $\langle \hat{\phi}^2 \rangle_{ad2} = \langle \hat{\phi}^2 \rangle^{(0)} + \langle \hat{\phi}^2 \rangle^{(2)}$, where again the superscripts indicate the adiabatic order.

We now compute the zeroth adiabatic order of $\langle \hat{\phi}^2 \rangle$ and regularize it by using the fact that the integral of a total derivative vanishes in dimensional regularization [14]. For this, and in order to avoid the complications of computing all quantities in n-dimensions, we first perform the angular integrations and then generalize the four-dimensional integrals to n-dimensions by replacing $d^3k = C^{3/2} d^3y = C^{3/2} y^2 dy d\Omega$ ($y_i = k_i/\sqrt{C_i}$) by $C^{3/2} y^{(n-2)} dy d\Omega$.

Therefore, the zeroth adiabatic order is given by

$$\langle \hat{\phi}^2 \rangle^{(0)} = \frac{1}{(2\pi)^3} \int y^{n-2} dy d\Omega \frac{1}{2\tilde{\omega}_k} = \frac{I_1}{2(2\pi)^2}, \tag{23}$$

where I_1 is given Table I. Note that the integral I_1 is divergent unless ω_k^2 behaves as x^s with $s > 3$, for large values of x. This divergence can be absorbed in the bare mass of the quantum field (see below).

$I_0 = \int_0^\infty dx\, x^{\frac{(n-3)}{2}} \tilde{\omega}_k$	$I_3 = \int_0^\infty dx \frac{x^{\frac{(n-3)}{2}}}{\tilde{\omega}_k^3}$
$I_1 = \int_0^\infty dx \frac{x^{\frac{(n-3)}{2}}}{\tilde{\omega}_k}$	$I_4 = \int_0^\infty dx \frac{x^{\frac{(n+1)}{2}}}{\tilde{\omega}_k^5} \frac{d^2 \tilde{\omega}_k^2}{dx^2}$
$I_2 = \int_0^\infty dx \frac{x^{\frac{(n+1)}{2}}}{\tilde{\omega}_k^3} \frac{d^2 \tilde{\omega}_k^2}{dx^2}$	$I = \int_0^{+\infty} dx \frac{x^{\frac{(n+3)}{2}}}{\tilde{\omega}_k^3} \frac{d^3 \tilde{\omega}_k^2}{dx^3}$

TABLE I: Explicit expressions for I_i. To obtain these integrals we have made the change of variables $x = y^2$ and we have defined $\tilde{\omega}_k = \omega_k/\sqrt{C}$.

The second adiabatic order can be written as

$$\langle \hat{\phi}^2 \rangle^{(2)} = -\frac{\sqrt{C}}{32\pi^3} \int y^{n-2} dy d\Omega \frac{{}^{(2)}W_k^2}{\omega_k^3}. \tag{24}$$

The angular integrations can be performed with the use of the identities listed in the Appendix A. After some calculations we obtain:

$$\begin{aligned}\langle\hat{\phi}^2\rangle^{(2)} &= -\frac{1}{16\pi^2}\left\{I_3\left[\frac{D^2}{16C}-\frac{D'}{4C}+R\left(\xi-\frac{1}{6}\right)+\frac{Q}{C}+\xi_2(\nabla_\mu u^\mu)^2+\xi_3\nabla_\mu u_\nu\nabla^\nu u^\mu+12\lambda\phi_0^2\right]\right.\\ &- \left.\left[\frac{3D^2}{8C}+2\frac{Q}{C}-\frac{D'}{4C}\right](J_{1000}+I_3)+\left[\frac{D^2}{16C}+\frac{Q}{5C}\right](J_{2000}+6J_{1000}-4J_{0100}+5I_3)\right\}, \qquad (25)\end{aligned}$$

where I_3 is given in Table I, and we have defined the integrals

$$J_{mnls} \equiv \int_0^\infty dx \frac{x^{\frac{(n-3)}{2}}}{\tilde{\omega}_k^3} f^m \, \dot{f}^n \, \ddot{f}^l \, \dddot{f}^s, \qquad (26)$$

with m, n, l, s integer numbers. As it is shown in the Appendix of Ref.[6], this integrals can be expressed in terms of the ones in Table I by performing integrations by parts. For $n \to 4$, we have [6]:

$$J_{1000} = 0, \qquad (27a)$$

$$J_{2000} = \frac{2}{5} I_4 \qquad (27b)$$

$$J_{0100} = \frac{3}{5} I_4. \qquad (27c)$$

Then, substituting this results into Eq. (25) we arrive at

$$\begin{aligned}\langle\hat{\phi}^2\rangle^{(2)} &= -\frac{I_3}{16\pi^2}\left[R\left(\xi-\frac{1}{6}\right)+12\lambda\phi_0^2+\xi_2(\nabla_\mu u^\mu)^2+\xi_3\nabla_\mu u_\nu\nabla^\nu u^\mu\right]\\ &+ \frac{I_4}{480\pi^2}\left[(\nabla_\mu u^\mu)^2+2\nabla_\mu u_\nu\nabla^\nu u^\mu\right]. \qquad (28)\end{aligned}$$

This is the main result of this section. The relevant point is that, in addition of the usual terms proportional to R and ϕ_0^2, the second adiabatic order contains terms with two derivatives of the aether field, which are present even if $\xi_2 = \xi_3 = 0$. For the standard dispersion relation I_3 diverges and I_4 vanishes (see Table I). Therefore, when $\xi_2 = \xi_3 = 0$ one reobtains the usual result. However, for any other dispersion relation of the type given in Eq. (9), I_3 and I_4 are finite. An interesting point is that, if we consider a generalized dispersion relation, evaluate the integral explicitly in four dimensions and *then* take the limit in which the dispersion relation tends to the usual one, a nonvanishing finite result can be obtained. For example, a dispersion relation of the form $\omega_k^2 = C(x + 2b_{11}x^2)$ yields

$$I_4 = 2b_{11}\int_0^{+\infty} dx(1+2b_{11}x)^{-\frac{5}{2}} = \frac{4}{3}. \qquad (29)$$

Therefore, there is a finite remnant of the trans-Planckian physics in the second adiabatic order, even in the limit in which the scale of new physics is very high $M_C \to \infty$ ($b_{11} \to 0$).

Coming back to the mean value equation (10), we write the bare parameters in terms of the renormalized ones plus the corresponding to counterterms:

$$\begin{aligned}&\Box\phi_0 - \Big[m_R^2+\delta m^2+(\xi_R+\delta\xi)R+(\xi_{2R}+\delta\xi_2)(\nabla_\mu u^\mu)^2+(\xi_{3R}+\delta\xi_3)\nabla_\mu u_\nu\nabla^\nu u^\mu\\ &+2\sum_{s,p} b_{sp}\mathcal{D}^{2(s+p)}+12\lambda_R\left(\langle\hat{\phi}^2\rangle_{ren}+\langle\hat{\phi}^2\rangle_{ad2}\right)\Big]\phi_0-4(\lambda_R+\delta\lambda)\phi_0^3=0. \qquad (30)\end{aligned}$$

Introducing Eqs. (23) and (28) into Eq. (30), we see that the regularized second adiabatic order $\langle\hat{\phi}^2\rangle_{ad2}$ can be absorbed into the bare constants by defining counterterms such that

$$\delta m^2 = -\frac{6\lambda_R}{(2\pi)^2} I_1, \tag{31a}$$

$$\delta\lambda = \frac{9\lambda_R^2}{(2\pi)^2} I_3, \tag{31b}$$

$$\delta\xi = \frac{3\lambda_R}{(2\pi)^2}\left(\xi_R - \frac{1}{6}\right) I_3, \tag{31c}$$

$$\delta\xi_2 = \frac{3\lambda_R\xi_{2R}}{(2\pi)^2} I_3 - \frac{\lambda_R}{40\pi^2} I_4, \tag{31d}$$

$$\delta\xi_3 = \frac{3\lambda_R\xi_{3R}}{(2\pi)^2} I_3 - \frac{\lambda_R}{20\pi^2} I_4. \tag{31e}$$

Note that, even when the parameters ξ_{2R} and ξ_{3R} are set to zero, the corresponding counterterms arise due to the self-interaction of the scalar field. Note also that by considering the same theory but in a background flat FRW space-time, it is not possible to distinguish between the redefinitions of ξ_2 and ξ_3 proportional to I_4, since in such background we have that

$$\nabla_\mu u_\nu \nabla^\nu u^\mu = \frac{1}{3}(\nabla_\mu u^\mu)^2. \tag{32}$$

From the results of this section we conclude that, as long as one considers the renormalization of the mean value equation in Bianchi type I spacetimes, and for the class of MDR considered here, it is enough to subtract the zeroth adiabatic order of $\langle\hat{\phi}^2\rangle$, since the second adiabatic order produce a finite renormalization of the bare constants of the theory. It would be interesting to check whether this is a general property, i.e. valid for an arbitrary background, or not. In order to address this issue, it would be necessary to know the singularity structure of the two-point function of a quantum field satisfying MDR for arbitrary values of $g_{\mu\nu}$ and u_μ. This singularity structure could be revealed by a generalized momentum-space representation of the Green's functions [15, 16]. In any case, the calculation of the second adiabatic order presented in this section shows that the interaction terms proportional to ξ_2 and ξ_3 that appear in Eq. (7) are generated by quantum effects, even if not present at the classical level. It is likely that the other interaction terms will also be generated in a more general background.

IV. ON THE RENORMALIZATION OF THE STRESS TENSOR IN BIANCHI TYPE I SPACE-TIMES

In this section we focus on the renormalization of the SEAE. We restrict the analysis to the case of a free scalar field ($\lambda = 0$, $\langle\phi\rangle = 0$) and, for the sake of simplicity, we set the parameters $\xi_i = 0$ (i=1,2,3,4,5).

The SEAE take the form

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G[T^{u_b}_{\mu\nu} + \langle T^{\tilde{\lambda}_c}_{\mu\nu} + T^{\phi}_{\mu\nu}\rangle + T^{clas}_{\mu\nu}] \tag{33}$$

where Λ and G are the bare cosmological and Newton's constants, $G_{\mu\nu}$ is the Einstein tensor, $T^{u,(\phi)}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S^{u(\phi)}}{\delta g^{\mu\nu}}$, and $T^{u}_{\mu\nu} = T^{u_b}_{\mu\nu} + T^{\tilde{\lambda}_c}_{\mu\nu}$, $T^{u_b}_{\mu\nu}$ is the stress tensor of the background vector field while $T^{\tilde{\lambda}_c}_{\mu\nu}$ is the additional contribution due to the modification of the Lagrange multiplier $\tilde{\lambda}$ arising from the coupling between the scalar field ϕ and u_μ. $T^{clas}_{\mu\nu}$ is a stress tensor coming from classical sources not coupled to the aether field. As we will compute the mean value of the stress tensor up to the second adiabatic order, we omit classical terms quadratic in the curvature (we will comment on this issue in the next section).

The nontrivial components of the Einstein tensor are, in Bianchi type I spacetimes:

$$G_{\eta\eta} = 3\left[\frac{D^2}{4} - Q\right], \tag{34a}$$

$$G_{ii} = -\frac{C_i}{2C}\left[3D' + \frac{3}{2}D^2 + 6Q - d_i' - d_i D\right]. \tag{34b}$$

The stress tensor corresponding to the background vector field u_μ can be written as

$$T^{u_b}_{\mu\nu} = -\frac{b_3}{8\pi G}G_{\mu\nu} - \frac{b_2}{8\pi G}\tilde{T}^{u}_{\mu\nu}, \tag{35}$$

whose nonzero components are

$$\tilde{T}^{u}_{\eta\eta} = \frac{9}{8}D^2, \tag{36a}$$

$$\tilde{T}^{u}_{ii} = -\frac{3}{2}\frac{C_i}{C}\left[D' + \frac{D^2}{4}\right]. \tag{36b}$$

The expectation value of the quantum energy momentum tensor $T_{\mu\nu} = T^{\phi}_{\mu\nu} + T^{\tilde{\lambda}_c}_{\mu\nu}$ is given by

$$\begin{aligned} \langle T_{\eta\eta}\rangle &= \frac{1}{2C}\int\frac{d^3k}{(2\pi)^3}\left\{|\chi_k'|^2 + 3D\left(\xi - \frac{1}{6}\right)(\chi_k'\chi_k^* + \chi_k{\chi_k'}^*) \right. \\ &+ \left. |\chi_k|^2\left[\omega_k^2 - 3D^2\left(\xi - \frac{1}{12}\right) + 2\xi G_{\eta\eta}\right]\right\}, \end{aligned} \tag{37}$$

$$\begin{aligned} \langle T_{ii}\rangle &= \frac{C_i}{C^2}\int\frac{d^3k}{(2\pi)^3}\left\{\left(\frac{1}{2} - 2\xi\right)|\chi_k'|^2 + \left(\frac{\xi}{2}(2D + d_i) - \frac{D}{4}\right)(\chi_k'\chi_k^* + \chi_k{\chi_k'}^*) \right. \\ &- \left. \xi(\chi_k''\chi_k^* + \chi_k{\chi_k''}^*) + |\chi_k|^2\left(k_i^2\frac{d\omega_k^2}{dk_i^2} - \frac{\omega_k^2}{2} + \frac{D^2}{8}\right) + \frac{\xi}{2}|\chi_k|^2\left[2D' - d_i D + 2\frac{C}{C_i}G_{ii}\right]\right\}. \end{aligned} \tag{38}$$

Using the expression given in Eq. (15) for the modes χ_k, it can be written as

$$\begin{aligned} \langle T_{\eta\eta}\rangle &= \frac{1}{2C}\int\frac{d^3k}{(2\pi)^3}\left\{\frac{[(W_k^2)']^2}{32W_k^5} - 3D\left(\xi - \frac{1}{6}\right)\frac{(W_k^2)'}{4W_k^3} + \frac{W_k}{2} \right. \\ &+ \left. \frac{1}{2W_k}\left[\omega_k^2 - 3D^2\left(\xi - \frac{1}{12}\right) + 2\xi G_{\eta\eta}\right]\right\}, \end{aligned} \tag{39}$$

$$\begin{aligned} \langle T_{ii}\rangle &= \frac{C_i}{C^2}\int\frac{d^3k}{(2\pi)^3}\left\{\left(\frac{1}{8} - 3\xi\right)\frac{[(W_k^2)']^2}{8W_k^5} + \xi\frac{(W_k^2)''}{4W_k^3} - \frac{(W_k^2)'}{4W_k^3}\left(-\frac{D}{4} + \frac{\xi}{2}(2D + d_i)\right) \right. \\ &+ \left. \frac{W_k}{4} + \frac{1}{2W_k}\left[k_i^2\frac{d\omega_k^2}{dk_i^2} - \frac{\omega_k^2}{2} + \frac{D^2}{8} + \xi D' - \frac{\xi}{2}Dd_i + \xi\frac{C}{C_i}G_{ii}\right]\right\}. \end{aligned} \tag{40}$$

Therefore, the zeroth adiabatic order can be expressed in the form

$$\langle T_{\eta\eta}\rangle^{(0)} = \frac{C}{2}\int\frac{d\Omega dy}{(2\pi)^3}y^{n-2}\tilde{\omega}_k = \frac{C}{2(2\pi)^2}\int_0^{+\infty}dx\,x^{\frac{n-3}{2}}\tilde{\omega}_k, \tag{41a}$$

$$\langle T_{ii}\rangle^{(0)} = \frac{C_i}{2}\int\frac{d\Omega dy}{(2\pi)^3}y^{n-2}\lambda_i^2\frac{y^2}{\tilde{\omega}_k}\frac{d\tilde{\omega}_k^2}{dy^2} = \frac{C_i}{3(2\pi)^2}\int_0^{+\infty}dx\,x^{\frac{n-1}{2}}\frac{d\tilde{\omega}_k}{dx}, \tag{41b}$$

where we have used that $\int d\Omega \lambda_i^2 = 4\pi/3$. Then, after an integration by parts in Eq. (41b) we obtain, as $n \to 4$,

$$\langle T_{\mu\nu}\rangle^{(0)} = -\frac{I_0}{2(2\pi)^2} g_{\mu\nu}, \tag{42}$$

where I_0 is a divergent integral as $n \to 4$ for any of the dispersion relations given in Eq. (9) (see Table I). Hence, this regularized adiabatic order can be absorbed into a redefinition of the bare cosmological constant Λ.

The second adiabatic order of $\langle T_{\mu\nu}\rangle$ can be written as

$$\langle T_{\eta\eta}\rangle^{(2)} = \frac{C}{2}\int \frac{d\Omega dy}{(2\pi)^3}\frac{y^{(n-2)}}{\tilde{\omega}_k}\left\{\frac{[(\omega_k^2)']^2}{32\omega_k^4} - 3D\left(\xi - \frac{1}{6}\right)\frac{(\omega_k^2)'}{4\omega_k^2} - \frac{3}{2}D^2\left(\xi - \frac{1}{12}\right) + \xi G_{\eta\eta}\right\}, \tag{43}$$

$$\begin{aligned}\langle T_{ii}\rangle^{(2)} &= C_i\int \frac{d\Omega dy}{(2\pi)^3}\frac{y^{(n-2)}}{\tilde{\omega}_k}\left\{\left(\frac{1}{8} - 3\xi\right)\frac{[(\omega_k^2)']^2}{8\omega_k^4} + \xi\frac{(\omega_k^2)''}{4\omega_k^2} - \frac{(\omega_k^2)'}{4\omega_k^2}\left(-\frac{D}{4} + \frac{\xi}{2}(2D + d_i)\right)\right.\\ &+ \left.\frac{{}^{(2)}W_k^2}{4}\left(1 - \lambda_i^2\frac{y^2}{\omega_k^2}\frac{d\omega_k^2}{dy^2}\right) + \frac{D^2}{16} + \frac{\xi}{2}D' - \frac{\xi}{4}Dd_i + \xi\frac{C}{2C_i}G_{ii}\right\},\end{aligned} \tag{44}$$

where ${}^{(2)}W_k^2$ is given by the expression in Eq. (19) with $\lambda = \xi_2 = \xi_3 = 0$. The explicit expressions for $(\omega_k^2)'/\omega_k^2$ and $(\omega_k^2)''/\omega_k^2$ are given in Eq. (21).

After performing the angular integrations with the use of the identities given in the Appendix A and some algebraic manipulations, we obtain:

$$\langle T_{\eta\eta}\rangle^{(2)} = \frac{1}{(2\pi)^2}[\alpha_1 D^2 + \alpha_2 Q], \tag{45a}$$

$$\langle T_{ii}\rangle^{(2)} = \frac{C_i}{C(2\pi)^2}[\beta_1 D^2 + \beta_2 D' + \beta_3 Dd_i + \beta_4 Q + \beta_5 d_i^2 + \beta_6 d_i']. \tag{45b}$$

The coefficients α_i and β_i are given in Appendix B, where it is also shown that using integration by parts they can be expressed in terms of two of the integrals in Table I. Thus, we find

$$\langle T_{\mu\nu}\rangle^{(2)} = \frac{1}{8\pi^2}\left\{\left[I_1\left(\xi - \frac{1}{6}\right) - \frac{I_2}{45}\right]G_{\mu\nu} + \frac{I_2}{30}\tilde{T}^u_{\mu\nu}\right\}. \tag{46}$$

Note that both I_1 and I_2 diverge when ω_k^2 behaves as $x^s, s \leq 3$ for large values of x. Therefore, in this case the divergences should be absorbed into the bare constants G and b_2. However, when $s > 3$, the second adiabatic order produce finite renormalizations of both constants.

As in the evaluation of $\langle\hat{\phi}^2\rangle$ presented in the previous section, depending on the dispersion relation one could have a remnant of the trans-Planckian physics in the second adiabatic order of $\langle T_{\mu\nu}\rangle$. Indeed, while I_2 vanishes for the standard dispersion relation, a non vanishing (and even divergent) result can be obtained for MDR in the limit $M_C \to \infty$. For example, for a dispersion relation of the form $\omega_k^2 = C(x + 2b_{22}x^4)$ we find that

$$I_2 = 24b_{22}\int_0^{+\infty} dx\frac{x^3}{(1 + 2b_{22}x^3)^{\frac{3}{2}}} = \frac{2^{\frac{8}{3}}}{\sqrt{\pi}b_{22}^{\frac{1}{3}}}\Gamma[1/6]\Gamma[4/3], \tag{47}$$

which diverges as $M_C \to \infty$ ($b_{22} \to 0$).

Eq. (46) is the main result of this section. We see that, for a generalized dispersion relation of the type given in Eq. (9), not only a redefinition of the Newton's constant is necessary in order to cancel the divergences of the second adiabatic order, but also a redefinition of the coefficient b_2 which corresponds to

the term $(\nabla_\mu u^\mu)^2$ in the bare Lagrangian of the vector field. The second adiabatic order contains terms that are non-purely geometric, in the sense that they cannot be written only in terms of the metric, but also involve the aether field.

It is noteworthy that for a background flat FRW space-time $G_{\mu\nu} = 3/2\tilde{T}^u_{\mu\nu}$, thereby, in Refs.[4, 6] it was not possible to realize that a redefinition of the Newton's constant is not enough for cancelling the second adiabatic order. In fact, for this particular space-time $G_{\mu\nu}$ is the unique covariantly conserved tensor of adiabatic order two that can be derived from an action formed by combining the vector field u^μ, the metric $g_{\mu\nu}$, and their derivatives.

V. DISCUSSION

In this paper we have worked within the context of a generally covariant theory of gravitation coupled to a dynamical time-like Lorentz-violating vector field. We considered a quantum scalar field satisfying MDR, and analyzed the renormalization of the infinities that arise in the semiclassical theory. In particular, considering Bianchi type I spacetimes, we have analyzed the dynamical equation for the expectation value of a self-interacting scalar field (Section III), and the SEAE for the metric in the case of a free scalar field (Section IV). With the use of adiabatic subtraction and dimensional regularization, we have shown that, in addition to the usual terms required to absorb the infinities of the second adiabatic orders, it is necessary to consider more general counterterms that involve the aether field. This property was not apparent in our previous works [4, 6], due to the high symmetry of the flat Robertson Walker metrics.

These results suggest that, in a more general background metric, any covariant term which can be formed by combining the vector field u^μ, the metric $g_{\mu\nu}$ and up to two of their derivatives, will appear in the regularized second adiabatic order of the expectation value of the quantum stress tensor, provided that the theory contains a scalar field with a generalized dispersion relation of the type given in Eq. (9). Hence, in order to absorb the divergences contained in the second adiabatic order, a bare action as general as the one given in Eq. (1) should be considered. Depending on the particular dispersion relation of the quantum field, the second adiabatic order may be finite. If this is the case, quantum effects generate finite renormalizations of the constants appearing in the classical Lagrangian. As we have also pointed out in Section IV, this finite renormalizations could be extremely large.

In the weak-field limit, the terms proportional to the constants b_i in Eq. (2) could have observable consequences. Indeed, the most general action given in Eq. (1) has four free parameters more than general relativity. This theory has been studied in several contexts, such as of the static weak-field limit [11], the radiation and propagation of the aether-gravitational waves [17], cosmology [18], etc., in which stringent constraints on the parameters have been imposed to make the theory consistent with observation. For example, in Ref. [11] it is shown that for all the PPN parameters to agree with observation, the four additional parameters of the model must satisfy two constraint equations with sufficient accuracy (i.e., the additional four-parameter space of the model has to be practically reduced to a two-dimensional subspace). In the absence of a known mechanism to explain why the parameters satisfy precisely such constraint equations, it seems that quantum effects generate a fine-tuning problem in the Einstein-Aether theory. This is analogous to the fine-tuning problem present in the Myers-Pospelov modification of QED [19].

In this paper we restricted ourselves to the evaluation of the adiabatic expansion up to the second adiabatic order, in a particular class of background metrics. By power counting, we expect the fourth adiabatic order $\langle T_{\mu\nu}\rangle^{(4)}$ to be finite in these metrics. However, there could be some subtleties related to the would be Gauss-Bonnet invariant in four dimensions [5, 6]. In the light of the results obtained in this paper, this issue should be reexamined. One should compute $\langle T_{\mu\nu}\rangle^{(4)}$ for a general background metric and aether field (this could be done by generalizing the momentum-space representation of the Green's functions). On dimensional grounds we expect $\langle T_{\mu\nu}\rangle^{(4)}$ to contain terms proportional to the variation of $R(\nabla_\mu u^\mu)^2$, $(R_{\mu\nu}u^\mu u^\nu)^2$, $R_{\mu\nu}u^\mu u^\nu R$, $R_{\mu\nu\rho\sigma}u^\mu u^\rho R^{\nu\sigma}$, etc., in addition to the usual ones: R^2, $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. Depending on the MDR, the fourth adiabatic order could be finite or divergent when expressed in terms of such variations. This fact would define whether it is necessary or not to subtract the fourth adiabatic order

in a general background, for a given dispersion relation. Work in this direction is in progress.

Appendix A: Identities for Bianchi type I space-times

In this Appendix we briefly summarize some useful formulas required for the adiabatic regularization of $\langle\hat{\phi}^2\rangle^{(2)}$ and $\langle T_{\mu\nu}\rangle^{(2)}$ in Bianchi type I space-times.

As we have already mentioned in the text, in order to regularize the theory we perform the four-dimensional angular integrations and then generalize the integrals to n-dimensions. We rescale the integration variables $k_i \to y_i = k_i/C_i$ and transform the volume element d^3y from rectangular coordinates to spherical coordinates $y^2dyd\Omega$, where $d\Omega$ is the solid angle element. In terms of $y_i^2 = y^2\lambda_i^2$, the relevant integrals are of the form

$$I(i,j,k) = \int d\Omega \lambda_1^{2i}\lambda_2^{2j}\lambda_3^{2k}, \tag{48}$$

which can be evaluated by using the fact that they are invariant under permutations of $\{i,j,k\}$. We provide here a list of the integrals we have used in this paper (see [20] for more details):

$$I(0,0,k) = \frac{4\pi}{2k+1}, \tag{49a}$$

$$I(1,1,0) = \frac{4\pi}{5\times 3}, \tag{49b}$$

$$I(1,2,0) = \frac{4\pi}{7\times 5}, \tag{49c}$$

$$I(1,1,1) = \frac{4\pi}{7\times 5\times 3}. \tag{49d}$$

These results, together with the formula $\sum_{i=1}^{3} d_i^2 = 3(8Q + D^2)$, allow us to derive the following identities:

$$\sum_{j=1}^{3}\sum_{k=1}^{3}\int d\Omega d_j d_k \lambda_j^2\lambda_k^2 = 4\pi\left(D^2 + \frac{16}{5}Q\right), \tag{50a}$$

$$\sum_{j=1}^{3}\sum_{k=1}^{3}\int d\Omega d_j d_k \lambda_i^2\lambda_j^2\lambda_k^2 = \frac{4\pi}{7\times 5}\left[5D^2 + 4d_iD + \frac{8}{3}d_i^2 + 16Q\right], \tag{50b}$$

$$\sum_{j=1}^{3}\int d\Omega(d_j' + 2d_jD - d_j^2)\lambda_i^2\lambda_j^2 = \frac{4\pi}{5\times 3}\left[2(d_i' + 2d_iD - d_i^2) + 3(D' + D^2 - 8Q)\right], \tag{50c}$$

that are useful for the evaluation of $\langle\hat{\phi}^2\rangle^{(2)}$ and $\langle T_{\mu\nu}\rangle^{(2)}$.

Appendix B: Regularization of $\langle T_{\mu\nu}\rangle^{(2)}$ in Bianchi type I space-times

In this Appendix we provide some details for computing the second adiabatic order of the expectation value of the quantum energy momentum tensor. The explicit expressions for the coefficients appearing in

Eq. (45)are:

$$\alpha_1 = \frac{1}{64}[-4I_{10} + I_{20} + 4(I_{00} - 6\xi I_{00} + 6\xi I_{10})], \tag{51a}$$

$$\alpha_2 = \frac{1}{20}[I_{00} + 2I_{10} + I_{20} - 30\xi I_{00}], \tag{51b}$$

$$\beta_1 = \frac{1}{2240}[52I_{00} - 120I_{01} - 76I_{10} + 20I_{11} + 77I_{20} - 5I_{30} + 280\xi(-I_{00} + 2I_{01} + I_{10} - I_{20})], \tag{51c}$$

$$\beta_2 = \frac{1}{640}[2I_{00} + 39I_{10} - 8I_{20} - 20\xi(22I_{00} + I_{10})], \tag{51d}$$

$$\beta_3 = \frac{1}{1680}[-8I_{00} + 12I_{01} - 19I_{10} + 12I_{11} - 14I_{20} - 3I_{30} + 210\xi I_{10}], \tag{51e}$$

$$\beta_4 = \frac{1}{140}[-24I_{01} + 3I_{10} + 4I_{11} + 21I_{20} - I_{30} - 14\xi(-8I_{01} + I_{10} + 4I_{20}) + I_{00}(42\xi - 19)], \tag{51f}$$

$$\beta_5 = \frac{1}{840}[2I_{00} + 4I_{01} + 3I_{10} + 4I_{11} - I_{30}], \tag{51g}$$

$$\beta_6 = \frac{1}{120}[-I_{00} - 2I_{10} - I_{20} + 30\xi I_{00}], \tag{51h}$$

where the integrals I_{mn} are given by

$$I_{mn} = \int_0^{+\infty} dx \frac{x^{\frac{n-3}{2}}}{\tilde{\omega}_k} f^m \dot{f}^n, \tag{52}$$

with $m, n = 0, 1, 2, 3$.

Let us now sketch the procedure to find relations between these coefficients in the context of dimensional regularization, which is completely analogous to the one described in the Appendix of Ref.[6] for relating the integrals J_{mnls} of Eq. (26). By definition, $I_{00} = I_1$, and

$$\begin{aligned} I_{10} &= \int_0^{+\infty} dx \frac{x^{\frac{n-3}{2}}}{\tilde{\omega}_k}\left(\frac{x}{\tilde{\omega}_k^2}\frac{d\tilde{\omega}_k^2}{dx} - 1\right) = -2\int_0^{+\infty} dx x^{\frac{n-1}{2}} \frac{d\tilde{\omega}_k^{-1}}{dx} - I_1 \\ &= (n-1)I_1 - I_1 \xrightarrow[(n\to 4)]{} 2I_1, \end{aligned} \tag{53}$$

where we have performed an integration by parts and discarded the surface term. Similarly, one can prove that

$$I_{20} = \frac{2}{3}I_2, \tag{54a}$$

$$I_{30} = \frac{4}{5}I_2 + \frac{8}{15}I, \tag{54b}$$

$$I_{01} = -2I_1 + \frac{1}{3}I_2, \tag{54c}$$

$$I_{11} = -\frac{2}{15}I_2 + \frac{2}{15}I, \tag{54d}$$

where the integrals I_i on the right hand side are given in Table I, and I (which does not appear in the final results) is given by

$$I = \int_0^{\infty} dx \frac{x^{\frac{(n+3)}{2}}}{\tilde{\omega}_k^3}\frac{d^3\tilde{\omega}_k^2}{dx^3}. \tag{54e}$$

Replacing these results into Eq. (51), we obtain:

$$\alpha_1 = \frac{3}{8} I_1 \left(\xi - \frac{1}{6}\right) + \frac{I_2}{96}, \tag{55a}$$

$$\alpha_2 = - \frac{3}{2} I_1 \left(\xi - \frac{1}{6}\right) + \frac{I_2}{30}, \tag{55b}$$

$$\beta_1 = - \frac{3}{8} I_1 \left(\xi - \frac{1}{6}\right) + \frac{I_2}{480}, \tag{55c}$$

$$\beta_2 = - \frac{3}{4} I_1 \left(\xi - \frac{1}{6}\right) - \frac{I_2}{120}, \tag{55d}$$

$$\beta_3 = \beta_6 = \frac{1}{4} I_1 \left(\xi - \frac{1}{6}\right) - \frac{I_2}{180}, \tag{55e}$$

$$\beta_4 = - \frac{3}{2} I_1 \left(\xi - \frac{1}{6}\right) + \frac{I_2}{30}, \tag{55f}$$

$$\beta_5 = 0. \tag{55g}$$

Finally, after substituting these coefficients into Eq. (45) we arrive at Eq. (46).

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